Overview of History of Elliptic Curves and its use in cryptography

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Abstract— Elliptic curves occur first time in the work of Diophantus in second century A.D. Since then the theory of elliptic curves were studied in number theory. Till 1920, elliptic curves were studied mainly by Cauchy, Lucas, Sylvester, Poincare. In 1984, Lenstra used elliptic curves for factoring integers and that was the first use of elliptic curves in cryptography. Fermat's Last theorem and General Reciprocity Law was proved using elliptic curves and that is how elliptic curves became the centre of attraction for many mathematicians.

Properties and functions of elliptic curves have been studied in mathematics for 150 years. Use of elliptic curves in cryptography was not known till 1985. Elliptic curve cryptography is introduced by Victor Miller and Neal Koblitz in 1985 and now it is extensively used in security protocol.

Index Terms— Elliptic curve, cryptography, Fermat's Last Theorem.

Introduction

Elliptic curves and its properties have been studied in mathematics as pure mathematical concepts for long since second or third century A.C. but its use in cryptography is very recent. The name "elliptic" itself was given in nineteenth century, though it has been studied widely by many mathematicians. Use of elliptic curve in cryptography was not known till 1984. The first application in cryptography is found in integer factorization method by Lenstra. In 1985, Victor Miller and Neal Koblitz proposed completely different cryptographic use of elliptic curves. Elliptic curve cryptography (ECC) is public key cryptography. ECC is based on properties of a particular type of equation created from mathematical group. Equations based on elliptic curves have characteristic that is very valuable for cryptographic purpose. The main reason for attractiveness of ECC is the fact that there is no sub exponential algorithm known to solve the discrete logarithm problem on a properly chosen elliptic curve. This means that significantly smaller parameters can be used in ECC with equivalent level of security.

Elliptic curves are the basis for a relative new class of public key schemes. It is predicted that elliptic curves will replace many existing schemes in near future. It is fascinating to know the origin and development of elliptic curves and how it has been used in cryptography?

This paper throws light on historical background of elliptic curves and its use in mathematics as well as cryptography.

Prehistory of elliptic curves

Elliptic curve is a curve of the form $y^2 = p(x)$, where $p(x)$ is a cubic polynomial with no repeated roots. Elliptic curve appears first time in the work of Diophantus in second or third century A.D. Diophantus had no concept of analytic geometry or modern algebraic notations and certainly no idea about elliptic curves. But for the first time where the elliptic curve appears is in the book of Diophantus's "Arithmetica". The problem written by him related to elliptic curve in his book read as follows:

"To divide a given number into two numbers such that their product is cube minus its side". And the equation that Diophantus wrote is $Y(a - Y) = X^3 - X$ which is actually an elliptic curve in disguise. The way Diophantus solved the problem is as follows:

Consider the equation $Y(a - Y) = X^3 - X$

Set $a = 6$ and Subtract 9 from both the sides gives

$6Y - Y^2 - 9 = X^3 - X - 9$

Replace Y by $y+3$ and X by $-x$ gives $y^2 = x^3 - x + 9$ which is an elliptic curve.

Diophantus solved the problem for $a = 6$ by substituting $X = 3Y - 1$, ignoring the double root he obtained the solution $y = 26/27$ and $x = 17/9$. Therefore the two numbers are $y = 26/27$ and $a - y = 136/27$ and the product of these two numbers is $(17/9)^3 - (17/9)$.

The exact nature of what Diophantus accomplished in the section of his problem took over 1500 years to reveal itself completely.

Elliptic curves then occurs roughly in eighth century and Fibonacci made it famous in eleventh century. He encountered the problem as to find a rational number r such that both $r^2 - 5$ and

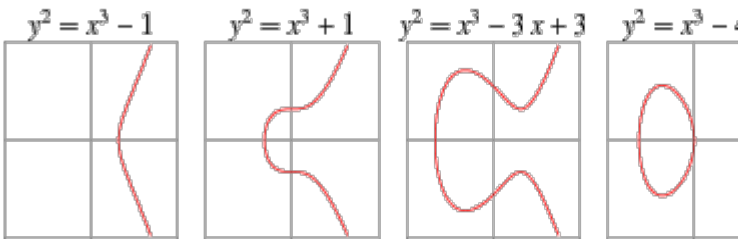
$r^2 + 5$ are rational squares. Fibonacci found such numbers namely, $r = 41/6$. Fibonacci called the positive integer "n" a congruent number if $r^2 - n$, r^2, $r^2 + n$ are all nonzero squares for some rational number r. The connection with elliptic curve is that if n is a congruent number then the product of the three nonzero rational squares $r^2 - n$, r^2, $r^2 + n$ is also a rational square. If we let $r^2 = x$, we get the equation $y^2 = x(x-n)(x+n)$, which represents elliptic curve. Thus if n is a congruent number, then the elliptic curve contains a nonzero rational point.

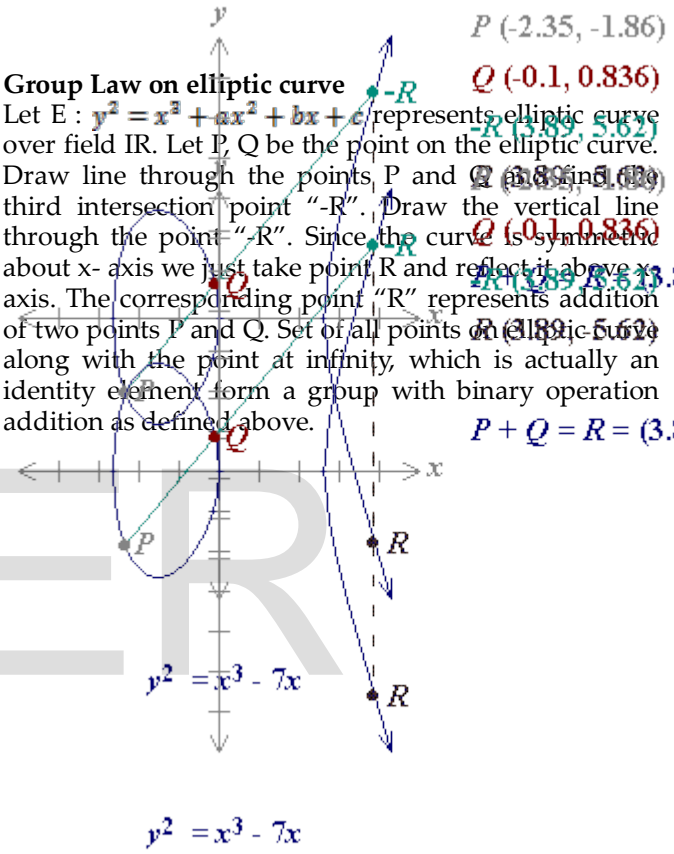
French mathematician Bachet made a Latin translation

of Diophantus's Arithmetica and published it in 1621.
Fermat acquired a copy of Arithmetica in 1630. Fermat's collected works contains several references to problems involving elliptic curves. In particular, his conjecture that the only integers satisfying the equation $y^2 = x^3 - 2$ are (x, y) = (3, 5) or (3, -5) and that the only integers satisfying $y^2 = x^3 - 4$ are (x, y) = (2, 2), (2, -2), (5, 11), (5, -11).
Euler obtained the copy of Fermat's work and he expanded the scope of number theory far beyond Fermat's work, he gave number theory its status as a legitimate field of mathematics. Euler also did quite a bit of work on congruent number problem and derived many results about elliptic integrals.
During 1670s Newton used recently developed tools of analytic geometry to classify cubic curves. In doing so, he explained the mysteries behind both Diophantus ' Arithmetica problem and Bachet's theorem about rational solution to elliptic curve.
In nineteenth century, Jacobi and Weirstrass connected these efforts with elliptic integrals and elliptic functions. In 1901, Poincare unified and generalized this work to algebraic curve.
The name "elliptic" is given because of the fact that these curves arose in studying the problem of finding the arc length of an ellipse. If one writes down the integral which gives the arc length of an ellipse and makes elementary substitution, the integrand will involve the square root of a cubic polynomial which is named as elliptic curve.
The invention of integral calculus in 1660's provided the new tool for solving the question of finding arc length of an ellipse. The first attempt to solve the arc length of an ellipse involved series and not integrals. In 1669, Newton expressed the arc length of an ellipse as an infinite series. Euler in 1733 and Maclaurin in 1742 also gave the series expression. To understand why, let us investigate what actually the problem arises while finding arc length of an ellipse.

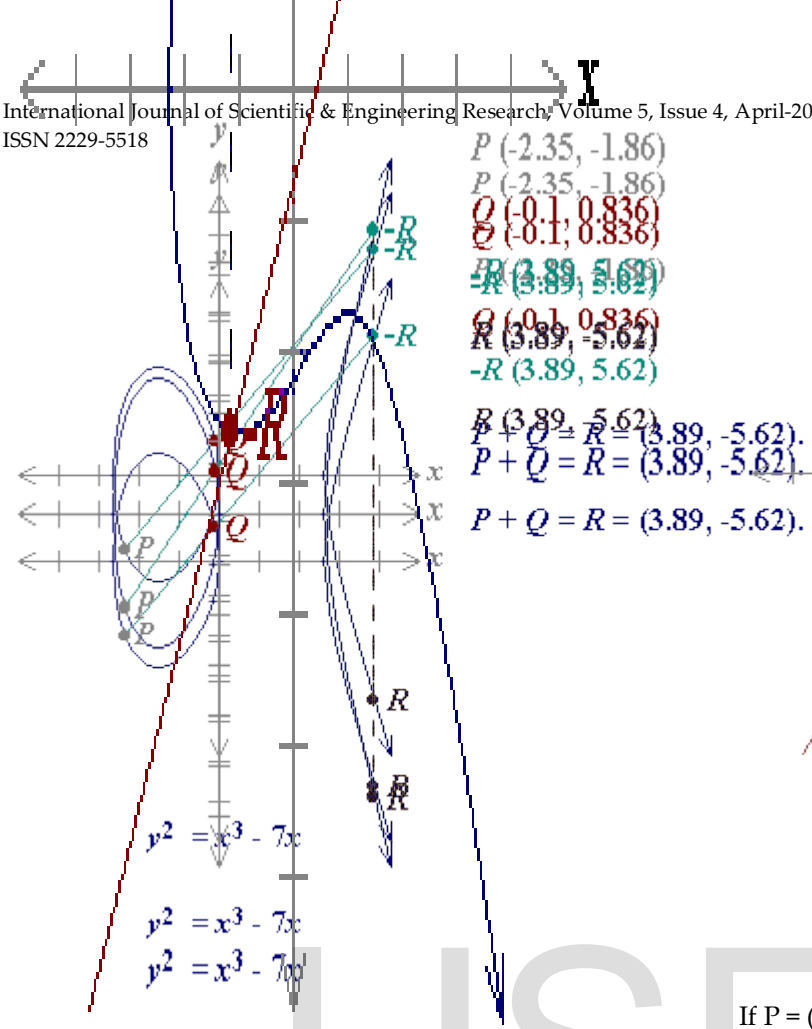
Arc length of an ellipse
If y = f(x) is continuous and has continuous derivative on the interval [a, b], then the arc length (L) of the curve is given by $L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx$.
If the curve is an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then with the parametrization $x = a\cos\theta, y = b\sin\theta$ and the substitution $1 - \frac{b^2}{a^2} = e^2$ gives $L = \int -a\frac{\sqrt{1-e^2x^2}}{\sqrt{1-x^2}} = \int -ay$ where $y^2(1-x^2) = 1 - e^2x^2$.
Again the substitution $u = \frac{1}{x}$ and $v = \frac{y(1-x)}{x}$ gives rise to an elliptic curve which is of the form
$v^2 = 2u^3(1-e^2) + u^2(5e^2 - 1) - 4e^2u + e^2$.

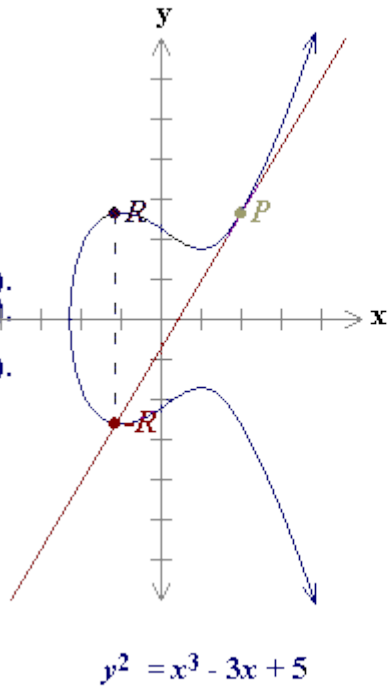
Definition of elliptic curve
An equation of the form $y^2 = x^3 + ax^2 + bx + c$ is called an elliptic curve. Some of the examples of elliptic curves are as follows:





Group Law on elliptic curve
Let E : $y^2 = x^3 + ax^2 + bx + c$ represents elliptic curve over field IR. Let P, Q be the point on the elliptic curve. Draw line through the points P and Q to find the third intersection point "-R". Draw the vertical line through the point "-R". Since the curve is symmetric about x- axis we just take point R and reflect it above x-axis. The corresponding point "R" represents addition of two points P and Q. Set of all points on elliptic curve along with the point at infinity, which is actually an identity element form a group with binary operation addition as defined above.

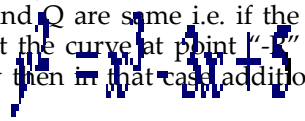




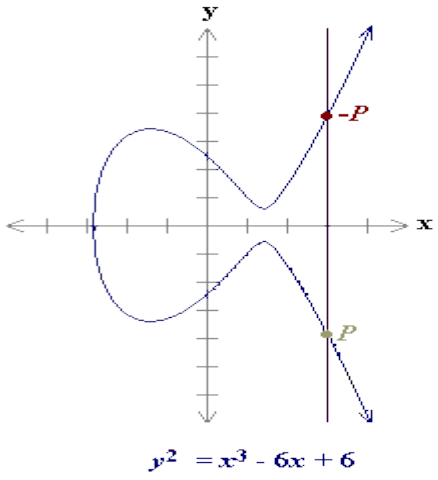
Thus if P = (x_1, y_1) and Q = (x_2, y_2) then P + Q = R = (x_3, y_3) is given by

$X_3 = [(y_2 - y_1)/(x_2 - x_1)]^2 - x_1 - x_2$.and $y_3 = -y_1 + [(y_2 - y_1)/(x_2 - x_1)](x_1 - x_3)$

If the points P and Q are same i.e. if the line through the point P meet the curve at point "-R" as shown in the figure below then in that case addition is taken as P+P = R.



If P = (x_1, y_1) and Q = (x_2, y_2) and $x_1 = -x_2$ then P + Q = P + (-P) is defined to be an identity element, which is point at infinity as shown below



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Use of elliptic curves in cryptography

The first use of elliptic curves in cryptography was Lenstra's elliptic curve factoring algorithm. This algorithm is a fast, sub exponential running time algorithm for integer factorization which employs elliptic curves. Lenstra invented new factorization

method using elliptic curves and it set a process of finding cryptographic uses that had never before being studied for this purpose. The largest factor found using elliptic curve factorization method so far has 83 digits and was discovered on September 7, 2013 by R. Propper.

In 1985, N. Koblitz and V. Miller independently proposed using the group of points on an elliptic curve defined over a finite field in discrete log cryptosystem. It is completely different way of solving cryptographic problems. One can use elliptic curve group that is smaller in size while maintaining the same level of security. In many situations, the result is smaller key size, bandwidth savings and faster implementation, especially in smart cards and cell phones.

In 2005, the U. S. National Security agency posted a paper in which they recommended that, "take advantages of the past 30 years of public key research and analysis and move from first generation public key algorithm on to elliptic curves.

Public key cryptography

Public key cryptography is a "one-way" mathematical process or function for which the inverse cannot feasibly be computed. In RSA system, the process is to take two very large randomly generated prime numbers and multiply them together. The inverse process is called integer factorization. In the Diffie-Hellman system, the operation is exponentiation in a finite field. The inverse of this process is called discrete logarithm in finite field.

Discrete logarithm problem

Let E: $y^2 = x^3 + ax^2 + bx + c$ represents elliptic curve over finite field. Let P, Q be points on elliptic curve. The problem is to find an integer k such that Q = KP.

Example

Let Consider an elliptic curve given by the equation $y^2 = x^3 + 9x + 17 \pmod{23}$.

Let P = (4, 5) and Q = (16, 5), Elliptic curve discrete logarithm problem is to find an integer k such that kP = Q.

The integer k can be found by repeated point doubling till we get Q.

Since P = (16, 5), 2P = (20, 20), 3P = (14, 14), 4P =(19, 20), 6P = (7, 3), 7P = (8, 7), 9P = (4, 5) = Q.

Thus 9P = Q and hence k = 9.

Solving discrete logarithm problem

At first the only algorithm known to solve the elliptic curve discrete log problems were generic one, that is they have nothing to do with specific structures of elliptic curve group. The first such algorithm designed in the setting of finite field discrete log by Pohling and Hellman. He uses Chinese remainder theorem to reduce discrete log problem in the prime order subgroup. This is why the groups of prime orders are usually chosen fir Diffie-Hellman type cryptosystem.

In a group G of prime order n, the two best generic algorithms, Baby step – Giant step and Pollards rho algorithm each requires running time roughly $O(\sqrt{n})$. Subsequently faster-than-square root algorithms were found for various classes of elliptic curves. However it still appears that the types of curves used in most cryptographic applications cannot be attacked by anything faster than the generic algorithms.

Conclusion

Jouney of elliptic curves since its inception is quite fascinating and its use in cryptography is amazing. After examining the security, implementation and performance of ECC applications, we can conclude that ECC is the most suitable public key cryptography scheme for use. Its efficiency and security makes it an attractive alternative to conventional cryptosystem. It is without a doubt, fast being recognized as a powerful cryptographic scheme.

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